BASIC METHODS FOR SOLVING EQUATIONS OF MATHEMATICAL PHYSICS

V.K. Andreev
Institute of Computational Modeling Russian Academy of Sciences, Russia

Keywords: Volume potential, simple layer potential, double layer potential, Green's formulae, Green's function, *a priori* estimates, fundamental solution, parametrix, eigenfunctions, Fourier method, Fourier series, Sturm-Liouville problem, integral transforms, Laplace transform, Fourier transform, Mellin transform, Hankel transform, Lie group, invariant, invariant and partially invariant solutions, conservation laws, approximate solutions, finite difference methods, grid method, finite element method, variational methods, projection methods, Ritz method, Galerkin method, Frechet derivative, Gateaux differential, Newton-Kantorovich method, monotone operators.

Contents

Summary

In mathematical physics models of various physical processes are studied. The basic mathematical objects are partial differential equations, integro-differential and integral equations. As a rule, a mathematical model has the form of a boundary value or initial boundary value problem for an equation or a system of equations. Elliptic, hyperbolic and parabolic equations define the major classes of these problems. We recognize classical and generalized (weak) formulations of these problems. An important concept of a generalized solution is based on the notion of a generalized derivative with the use of the Sobolev spaces. The theory of differential and integral equations, the theory of functions and embedding theorems, methods of functional analysis, approximate methods and computational mathematics are the main tools for studying problems of mathematical physics. In this chapter basic methods for solving problems of mathematical physics are presented: potential methods, eigenvalue problems, methods of integral transforms, methods of transformation groups (analytical methods); discretization methods, methods for nonlinear problems, variational methods (approximate methods).

Introduction

Active development of productive forces in the countries of Western Europe late in the 17th century and early in the 18th century became the main reason for significant progress in natural sciences and engineering, including celestial mechanics, hydrodynamics, dynamics of a system of material points and the theory of electricity, magnetism, and heat. On the basis of these sciences there arose directions of research and problems which subsequently led to basic problems, notions and theorems of mathematical physics. For example, two problems closely related to each other arose in celestial mechanics: the problem on the shape of the Earth and other planets and the problem on their mutual attraction. The former problem led to the complicated theory of the shape of a planet (this theory is important in navigation, geography and geodesy) and the latter provided the basis for the development of the theory of attraction which became the foundation for the theory of Newtonian potential. In 1686 Newton stated its basic principles in his famous “Mathematical Elements”. Further investigations concerned with attraction of spheroids are due to Laplace and Lagrange. In 1765 Euler introduced the notion of a potential for problems of hydrodynamics. The concept of a potential as a function whose gradient is a vector field is due to Gauss. Coulomb and Poisson were the first to study properties of a simple layer potential. An important contribution to the development of the potential theory was made by Green starting in 1828 when he began to study boundary value problems. However, the ideas of the potential method appeared to be very useful in the theory of elliptic and parabolic equations and of systems of a general form, in particular, when studying existence and smoothness of solutions of linear and nonlinear boundary value problems. These investigations were conducted by scientists of different countries in the second part of the 20th century and are still in progress.

A new stage of the development of mathematical physics began in the 20th century when problems of the theory of relativity, quantum mechanics, hypersonic gas dynamics, kinetic equations, the theory of nuclear reactors, plasma physics, biology, and economics were involved in this field of science. Many problems of mathematical physics are reduced to boundary value problems for differential (or integro-differential) equations which, combined with boundary (and initial) conditions, form mathematical models of physical processes. The theory of partial differential equations, the theory of integral equations, the theory of functions and functional analysis, approximate methods are basic mathematical tools for studying these problems.

By convention methods for solving problems of mathematical physics can be divided into two wide classes. *Analytical methods* that include construction and justification of exact solutions in the form of finite formulae or series, problems of existence, uniqueness, and smoothness of a solution, form the first class. The second class involves *approximate methods*: discretization methods, splitting methods, iterative methods for solving nonlinear problems.

1. Analytical Methods for Problems of Mathematical Physics

The term analytical methods implies the study and rigorous derivation of finite formulae and analysis of the inner mathematical structure of a model. In both cases there arise questions which lead to more general mathematical problems: existence and uniqueness of a solution, integral representation of a solution, representation of a solution as a series etc. These problems are not immediately related to some special model.

1.1. Methods of the Potential Theory

1.1.1. Green's Formulae

Green's formulae are used for calculation of an integral of a function of many variables and relate the value of an n-multiple integral over a domain $\Omega \in R^n$ to that of an $(n-1)$-multiple integral over a piecewise smooth boundary Γ of this domain.The simplest formula has the form

$$\int_{\Omega}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy=\int_{\Gamma}(P\,dx+Q\,dy),\qquad \Omega\subset R^2. \tag{1}$$

A domain Ω is assumed to be oriented in a natural way. Formula (1) (known to Euler in the 18th century) has a simple hydrodynamical sense: the flow of a liquid, moving over a plane with velocity $\mathbf{v}=(Q,-P)$ across a boundary Γ, is equal to the integral over a domain Ω of intensity (divergence) $\operatorname{div}\mathbf{v}=\partial Q/\partial x-\partial P/\partial y$ of sources and sinks distributed in Ω.

The following formulae are due to Green (1828):

$$\int\limits_{\Omega}\left(v\Delta u+\sum_{i=1}^{3}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial y}\right)dx=\int\limits_{\Gamma}v\frac{\partial u}{\partial N}\,d\Gamma \tag{2}$$

and

$$\int\limits_{\Omega}\left(v\Delta u-v\Delta u\right)dx=\int\limits_{\Gamma}\left(u\frac{\partial v}{\partial N}-\frac{\partial u}{\partial N}\right)d\Gamma. \tag{3}$$

Here $\Omega\subset R^3$, $x=(x_1,x_2,x_3)$, $dx=dx_1\,dx_2\,dx_3$, $N=(N_1,N_2,N_3)$ is a unit outer normal to Γ, $\partial/\partial N=\sum_{i=1}^{3}N_i\,\partial/\partial x_i$, $\Delta=\sum_{i=1}^{3}(\partial/\partial x_i)^2$ is the Laplace operator. Formulae (2) and (3) are also valid for $\Omega\subset R^n$.

The generalization of formulae (2) and (3) for linear partial differential operators, provided that their coefficients are sufficiently smooth, has the following form:

1) if

$$Lu=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)+\sum_{i=1}^{n}b_i(x)\frac{\partial u}{\partial x_i}+c(x)u,$$

$$L^{*}v=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial v}{\partial x_j}\right)+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(b_i(x)v)+c(x)v$$

are adjoint real operators of the second order, $a_{ij}=a_{ji}$, then

$$\int\limits_{\Omega}\left[vLu+\sum_{i,j=1}^{n}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}-\left(\sum_{i=1}^{n}b_i\frac{\partial u}{\partial x_i}+cu\right)v\right]dx=\int\limits_{\Gamma}v\frac{\partial u}{\partial M}\,d\Gamma,$$

$$\int\limits_{\Omega}\left(uL^{*}v-vLu\right)dx=\int\limits_{\Gamma}\left(u\frac{\partial v}{\partial M}-v\frac{\partial u}{\partial M}-Buv\right)d\Gamma,$$

where $B=\sum_{i=1}^{n}N_ib_i$, $\partial/\partial M=\sum_{i,j=1}^{n}a_{ij}N_j\,\partial/\partial x_i$ is a derivative with respect to a conormal,

$$M=\left(\sum_{j=1}^{n}a_{1j}\,\partial/\partial x_j,\ldots,\sum_{j=1}^{n}a_{nj}\,\partial/\partial x_j\right);$$

2) if

$$Lu=\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial^2 u}{\partial x_i\,\partial x_j}+\sum_{i=1}^{n}b_i(x)\frac{\partial u}{\partial x_i}+c(x)u,$$

$$L^*v = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i\,\partial x_j}(a_{ij}(x)v) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i(x)v) + c(x)v,$$

then

$$\int_\Omega \left[vLu + \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} - \left(\sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu \right) v \right] dx = \int_\Gamma v\frac{\partial u}{\partial M}\,d\Gamma,$$

$$\int_\Omega (uL^*v - vLu)\,dx = \int_\Gamma \left(u\frac{\partial v}{\partial M} - v\frac{\partial u}{\partial M} - Cuv \right) d\Gamma,$$

where M is a conormal and $C = \sum_{i=1}^{n} N_i \left(b_i - \sum_{j=1}^{n} \partial a_{ij}/\partial x_j \right)$;

3) if

$$Lu = \sum_{p=1}^{m} \sum_{|\alpha|=p} a_\alpha(x) D^\alpha u + a(x)u, \qquad L^*v = \sum_{p=1}^{m} (-1)^p \sum_{|\alpha|=p} D^\alpha (a_\alpha(x)v) + a(x)v$$

are adjoint real operators of order m, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_p)$ is an integer multi-index of length $|\alpha| = p$, $1 \le \alpha_i \le n$, $D^\alpha = \partial_{\alpha_1}\partial_{\alpha_2}...\partial_{\alpha_p}$, $\partial_i = \partial/\partial x_i$, then

$$\int_\Omega (uL^*v - vLu)\,dx = \sum_{p=1}^{m} \sum_{|\alpha|=p} \sum_{k=1}^{p} \int_\Gamma (-1)^k [\partial_{\alpha_1}...\partial_{\alpha_{k-1}}(a_\alpha v)]x \times N_{\alpha_k}[\partial_{\alpha_{k+1}}...\partial_{\alpha_p} u]\,d\Gamma. \qquad (4)$$

Here the integral over a boundary can be written as a bilinear form $\sum_{i,j}\int_\Gamma B_{ij}(S_i u)(T_j v)\,d\Gamma$ with some differential operators S_i, T_j of order s_i, t_j, $0 \le s_i + t_j \le m-1$, respectively.

Green's formulae play an important role in calculus and, especially, in the theory of boundary value problems for differential and partial differential operators of the second (and higher) order.

For functions $u(x), v(x)$ being sufficiently smooth in $\bar{\Omega} \cup \Gamma$, where $\bar{\Omega} = \Omega \bigcup \Gamma$, these formulae lead to some relations useful in studying properties of a solution of a boundary value problem, in determining the kind of a boundary value problem, in deriving a solution in an explicit form, and in establishing *a priori* estimates of a solution up to a boundary.

-
-
-

Bibliography

Agmon S., Douglis A., and Nirenberg L. (1959). *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions.* I. Comm. Pure Appl. Math. 12, 623-727. [*A priori* estimates for systems of elliptic equations with general boundary conditions are established.]

Andreev V.K., Kaptsov O.V., Pukhnachov V.V., and Rodionov A.A. (1988). *Applications of Group-Theoretic Methods in Hydrodynamics.* Kluwer Academic Publishers, Dordrecht-Boston-London. [The group classification of equations of hydrodynamics in the Lagrange coordinates is presented. New classes of exact solutions are constructed.]

Andreev V.K., Boublik V.V., and Bytev V.O. (2003). *Symmetry of Nonclassical Models in Hydrodynamics.* Nauka, Novosibirsk. [The concept of invariance in nonclassical models in hydrodynamics is presented.]

Arima R. (1964). *On general boundary value problem for parabolic equations.* J. of Math. of Kyoto Univ., 4, № 21, 207-244. [General boundary value problems for parabolic equations are studied with the help of operators of fractional differentiation.]

Bers L., John F., Schechter M., Garding L., and Milgram A.N. (1964). *Partial Differential Equations.* Interscience Publ., New York-London-Sydney. [The book covers the current state of the theory of partial differential equations.]

Bocharov A.V., Verbovetsky A.M., Vinogradov A.M. (1997). *Symmetries and conservation laws of mathematical physics equations.* Factorial, Moscow. [The foundations theory of higher and non-local symmetrics and conservation laws for general system of differential equations is constracted.]

Eudelman S.D. (1964). *Parabolic Systems.* Nauka, Moscow. [The theory of general boundary value problems for parabolic equations and systems is presented.]

Friedman A. (1964). *Partial Differential Equations of Parabolic Type,* Prentice-hall, Inc., Englewood Cliffs. [Initial boundary value problems for parabolic equations are studied with the help of the method of heat potential.]

John F. (1955). *Plane Waves and Spherical Means Applied to Partial Differential Equations.* Interscience Publ., New York. [Fundamental solutions of many elliptic equations are constructed.]

Kantorovich L.V. and Akilov G.P. (1977). *Functional Analysis.* Nauka, Moscow. [The theory of linear and nonlinear operators and the general theory of approximate methods in functional spaces are presented.]

Ladyzhenskaya O.A. (1970). *Mathematical Problems in Viscous Incompressible Fluid Dynamics.* Nauka, Moscow. [Existence of a solution of stationary and nonstationary problems for the Navier-Stokes equations is considered.]

Ladyzhenskaya O.A. (1973). *Boundary Value Problems of Mathematical Physics.* Nauka, Moscow. [The theory of weak solutions is presented.]

Lebedev N.N. (1963). *Special Functions and Their Applications.* Fizmatgiz, Moscow-Leningrad. [New integral transforms with kernels involving the McDonald function are considered.]

Litvinov V.G. (1982). *Motion of Nonlinearly Viscous Liquid.* Nauka, Moscow. [Existence and uniqueness of a solution of stationary and nonstationary problems on motion of nonlinearly viscous liquid are considered.]

Miller W. (1977). *Symmetry and Separation of Variables.* School of Mathematics University of Minnesota. [The relation between the Lie algebra of symmetries of an equation and a coordinate system, in which the equation admits separation of variables, is established.]

Miranda C. (1955). *Equazioni Delle Derivate Parziali di Tipo Ellitico.* Springer, Berlin. [Elliptic equations are studied by the method of integral equations.]

Olver P.J. (1989). *Applications of Lie Groups to Differential Equations.* Springer-Verlag; New York-Berlin-Heidelberg-Tokyo. [Many examples of the solution of various problems of mechanics and hydrodynamics by application of the theory of the Lie groups are considered. The Noether theory of conservation laws is presented.]

Ovsyannikov L.V. (1978). *Group Analysis of Differential Equation.* Nauka, Moscow. [The foundations of the theory of continuous transformation groups are presented. Many problems of group classification for various models of mechanics of a continuous medium are solved.]

Samarskii A.A. (1989). *Theory of Difference Schemes.* Nauka, Moscow. [Basic problems of the theory of difference schemes, which arise in the solution of equations of mathematical physics, are systematically presented.]

Samarskii A.A. and Goulin A.V. (2000). *Numerical Methods in Mathematical Physics.* Nauchnyi mir, Moscow. [The foundations of the theory of difference schemes and of the finite element method, principles of the construction and study of computational algorithms for problems of mathematical physics are presented.]

Schwartz L. (1957). *The'orie des Distributions.* Hermann, Paris. [The theory of distributions of functional analysis point of view is described.]

Sneddon I.N. (1972). *The Use Integral Transform.* McGraw-Hill, New York. [Almost all integral transforms with inversion formulae are presented.]

Uflyand Ya.S. (1967). *Integral Transforms in Problems of the Elasticity Theory.* Nauka, Leningrad. [Integral transforms, being generalization of the Kantorovich-Lebedev transform, are studied. Their applications to problems of the elasticity theory are considered.]

Biographical Sketch

V.K. Andreev was born in village Kalinino of Nerchinsk region, Russia. He completed his Diploma in Mechanics and Applied mathematics at the Novosibirsk State University in 1972. In 1975 he took the Russian degree of Candidate in Physics and Mathematics at the Institute of Mathematics of the USSR Academy of Sciences, Novosibirsk. In 1990 V.K. Andreev was awarded the Russian degree of Doctor in Physics and Mathematics from Institute of Hydrodynamics of the USSR Academy of Sciences, Novosibirsk with the thesis "Invariant solutions of the hydrodynamics equations with a free boundary and their stability". In 1992 he took Professor Diploma at the Chair "Mathematical analysis and differential equations". From 1975 to 1990 he was a senior worker at the Computer Center of the USSR Academy of Sciences in Krasnoyarsk, Russia (now renamed the Institute of Computational Modeling of the Russian Academy of Sciences). Since 1990 V.K. Andreev is Head of Department "Differential equations of mechanics" at the same institute. V.K. Andreev is a well-known specialist in the fields of computational hydrodynamics, group analysis and hydrodynamic stability. He authored five monographs and over 170 scientific papers, devoted mathematical modeling, numerical methods of solving boundary value problems, group classification and hydrodynamic stability.