FUZZY DECISION THEORY

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Contents

Summary

In real decision situations a decision maker (DM) is often confronted with the problem that the information which is necessary for constructing a classical decision model is not available, or the cost for getting this information seems too high. Subsequently, the DM abstains from constructing a decision model; the DM fears that this model is not an authentic image of the real problem. Fuzzy set theory offers the possibility to construct decision models with vague data. Many decision models with fuzzy components are proposed in literature, but only fuzzy consequences and fuzzy probabilities are important for practical applications. Therefore, the focus of this paper is concentrated on these subjects. It is shown that the principle of Bernoulli can easily be extended to decision models with fuzzy utilities. Furthermore it is possible to use additional information in order to improve the prior probabilities. Moreover, fuzzy probabilities can be used combined with crisp utilities, described by real numbers, or fuzzy utilities. Apart from the fact that fuzzy models offer a more realistic modeling of decision situations, the proposed interactive solution process leads to a reduction of information costs. That circumstance is caused by the fact that additional information is gathered in correspondence with the requirements and under consideration of cost–benefit relations.

1. Classical Decision Model

Looking at modern theories in management science and business administration, one recognizes that the majority of these conceptions are based on decision theory in the sense of von Neumann and Morgenstern. However, empirical surveys reveal that decision models are hardly used in practice to solve real-life problems. This neglect of recognized classical decision concepts may be caused by the fact that the information necessary for modeling a real decision problem is not available, or the cost for getting this information seems too high.

In order to design a decision problem by classical decision models, the decision maker (DM) must be able to specify the following elements:

1. A set A of actions, $A = \{a_1, a_2, \dots, a_m\}$,
2. A set S of possible events, $S = \{s_1, s_2, \dots, s_n\}$,
3. A result associated with each act-event combination, $g_{ij} = g(a_i, s_j)$, $i = 1, 2, \dots, m;$ $j = 1, 2, \dots, n$. G is the set of possible values g_{ij}.
4. The degree of knowledge with regard to the chance of occurrence of each event. Usually it is assumed that the DM knows the probability distribution $p(s_j)$.
5. A criterion by which a course of action is selected:
   In literature, the Bernoulli-criterion is recommended, i.e. the expected utility should be maximized:

$$E(a^*) = \max_{a_i \in A} E(a_i) = \max_{a_i \in A} \sum_{j=1}^{n} u(g(a_i, s_j)) \cdot p(s_j) \qquad (1)$$

6. A posteriori probability distribution:

   In classical decision models the only chance for getting a better solution is to use additional information $X = \{x_1, x_2, \dots, x_K\}$. Knowing the likelihoods $p(x_k \mid s_j)$, the priori probability distribution $p(s_j)$ can be substituted by the posteriori probability distribution

$$p(s_j \mid x_k) = \frac{p(x_k \mid s_j) \cdot p(s_j)}{\sum_{j=1}^{n} p(x_k \mid s_j) \cdot p(s_j)} \quad \text{Bayes's formula} \qquad (2)$$

With the additional information that x_k is observed the optimal action $a^*(x_k)$ satisfies the term

$$E(a^*(x_k)) = \max_{a_i \in A} \sum_{j=1}^{n} u(g(a_i, s_j)) \cdot p(s_j \mid x_k).$$

The expected value of additional information is

$$E(X) = \sum_{k=1}^{K} E(a^*(x_k)) \cdot p(x_k) - E(A^*) .$$

Therefore, additional information should be used if the costs for getting this information are smaller than E(X). (See Decision Making under Risk and Uncertainty, Decision Problems and Decision Models.)

2. Basic Definitions of the Fuzzy Set Theory

The foundations for the fuzzy set theory were laid by Lotfi A. Zadeh in his paper *Fuzzy Sets*. It is the most important contribution to "fuzzy" literature and the publication date 1965 marked the birth of the new mathematical discipline, the fuzzy set theory.

By means of the fuzzy set theory, more realistic mathematical models can be designed for real-world problems. Therefore this new theory has also been considered as a new way of modeling decision models. For presenting these new instruments some basic definitions are necessary.

A fuzzy set is a generalization of the classical notion of a set in the sense of Cantor: "A set is defined as the combination of special well distinguished objects of our recognition or our thinking as a whole."

The classic set is strongly limited. According to the two-valued logic, which only allows Yes/No-statements, each object has to be either an element of the set or not. This strong delimitation of a set, which is characterized as "crisp set," quite often causes difficulties in the case of application to real problems, as shown by the following example:

Among a set of people joining a family meeting we have to choose the subset A of "young men." If one of the persons present should choose the set A, it would be easy for that person to describe some persons as elements of A and others as not belonging to this group. But there would also be borderline cases where the membership is not obvious.

According to Zadeh for each element x of a universe set X, the grade of membership of a fuzzy subset Ã could be expressed as a set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$, where the membership function is defined as $\mu_{\tilde{A}} : X \to [0, 1]$.

Similar to the definition of Cantor, the functional value 0 is given to objects which definitely do not show the requested attributes.

If the value set $[0,1]$ implies that objects with the membership value 1 definitely belong to the required set, Zadeh's concept of a fuzzy set is directly an extension of the set definition by Cantor, where the value set is limited on the set $\{0, 1\}$. Sets in the sense of Cantor are called crisp sets.

The use of a numeric scale like the interval [0,1] provides a simple and clear presentation of the membership degrees. In order to avoid misinterpretation it is necessary to emphasize that the membership values are always the expression of the personal estimation of single persons or groups. In case of the example above, a 30-year-old woman will most probably fix other membership degrees than an 80-year-old man. Furthermore the membership values also depend on the universe set X.

Figure 1 shows a membership function of the fuzzy set “young men” based on an opinion poll among students of the RWTH Aachen. The membership function in Figure 1 is an continuous approximation to the observed points (age; number of nomination in relation to the maximal nomination of an age). Here the universe set is [0, 100], where the opinion poll was restricted to integer numbers.

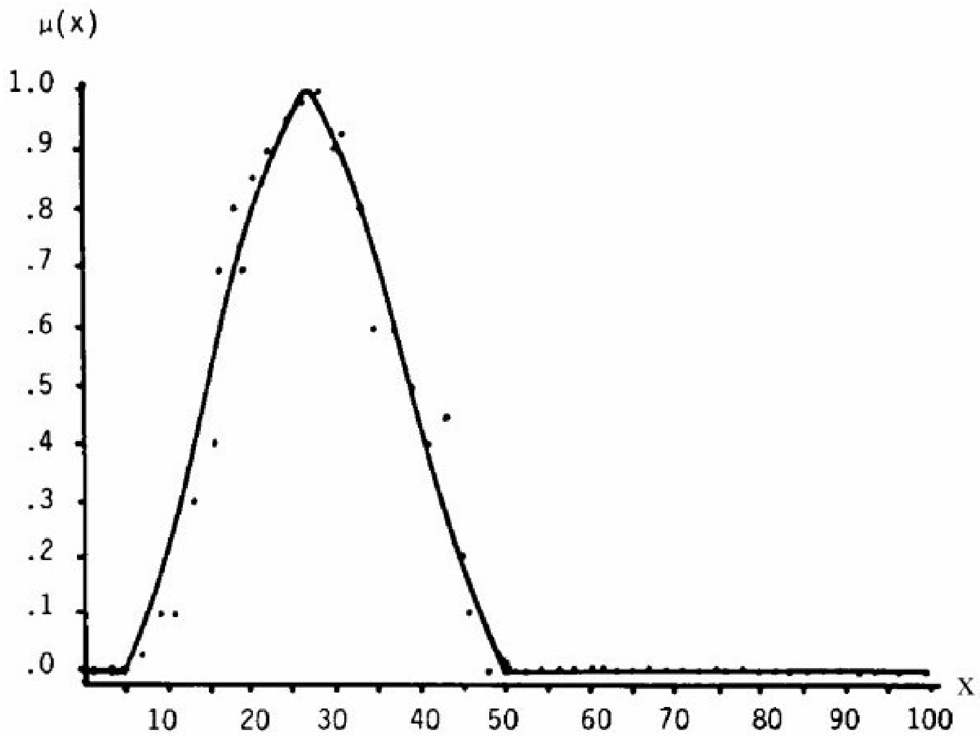


Figure 1. Membership function of the fuzzy set “young men”

According to the definition, a membership function $\mu_{\tilde{A}}$ maps the universe set X *in* the interval $[0,1]$ but not necessarily *on* the interval $[0,1]$. Since many applications for fuzzy sets are only useful, if the all membership functions have the same value set, it is required that all fuzzy sets are normalized by dividing its membership function $\mu_{\tilde{A}}$ by $\underset{x \in X}{Sup} \; \mu_{\tilde{A}}(x)$.

For describing subsets, it is often sufficient to look at the elements of X with positive membership degrees. The support (set) supp (Ã) = $\{x \in X \mid \mu_{\tilde{A}}(x) > 0\}$ is a crisp subset of X. In the special case that $\tilde{A}$ is a crisp set then supp (Ã) = $\{x \in X \mid \mu_{\tilde{A}}(x) = 1\}$

Analogously to the support, additional crisp sets may be used for describing fuzzy sets. For a given membership degree, the α-level set or α-cut of a fuzzy set $\tilde{A}$ is defined as: $A_\alpha = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\}$ where $\alpha \in [0,1]$. The importance of the α-cuts is obvious in the representation theorem; it says that a fuzzy set $\tilde{A}$ is completely characterized by the accompanying family of α-level-sets because the membership function of $\tilde{A}$ can be written as

$$\mu_{\tilde{A}}(x) = \sup\{\alpha \in [0,1] \mid x \in A_\alpha\} \qquad (3)$$

Therefore, an approximation of a fuzzy set can be constructed by using few α-cuts; see the construction of fuzzy intervals of the ε-λ-type in section 3.

Though the concept of fuzzy sets offers the possibility of describing the membership function of a fuzzy set in some detail, this is almost impossible in practical use and requires immense investigations. Therefore, the membership functions used in practice are in general a rough description of the subjective imagination. Quite often simple kinds of functions are used and special kinds of fuzzy sets are applied, namely fuzzy numbers and fuzzy intervals, which are directly the extension of the crisp terms (see Figure 2).

A fuzzy number is a convex normalized fuzzy set $\tilde{A}$ on the real line **R** such that

1. there exist exactly one real number x with a membership degree $\mu_{\tilde{A}}(x) = 1$,
2. $\mu_{\tilde{A}}(x)$ is piecewise continuous in **R**.

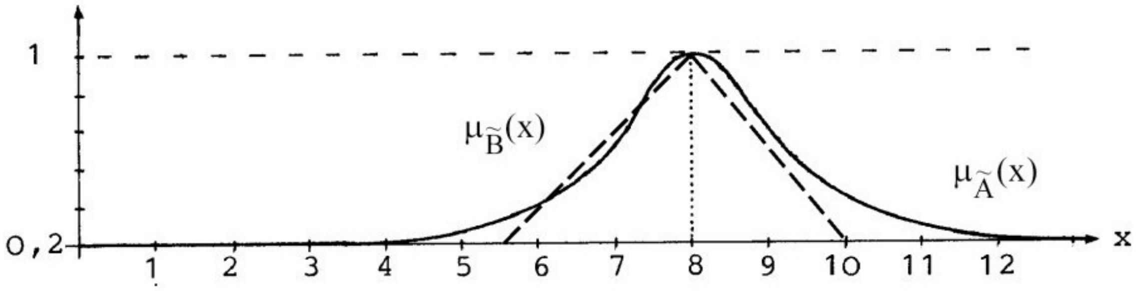
Thereby, a fuzzy set $\tilde{A}$ on a convex set X is called convex, if

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq Min(\mu_{\tilde{A}}(x_1),\, \mu_{\tilde{A}}(x_2)), \quad x_1, x_2 \in X, \quad \lambda \in [0,1] \qquad (4)$$

This characteristic has the consequence that all α-cuts A_α are crisp intervals on **R**.

Fuzzy intervals are extensions of fuzzy numbers where the 1-cuts are not restricted to one point but are extended to crisp intervals.

Figure 2 presents two examples for describing "approximately 8." As the membership function $\mu_{\tilde{B}}(x)$ resembles a triangle, this special fuzzy set is called triangular fuzzy number.



$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \in \mathbf{R} \times [0,1] \mid \mu_{\tilde{A}}(x) = (1 + (x-8)^2)^{-1}\}$$

$$\tilde{B} = \{(x, \mu_{\tilde{B}}(x)) \in \mathbf{R} \times [0,1]\} \text{ with } \mu_{\tilde{B}}(x) = \begin{cases} \dfrac{x-5{,}5}{1{,}5} & \text{for } 5{,}5 \le x < 8 \\ \dfrac{10-x}{2} & \text{for } 8 \le x \le 10 \\ 0 & \text{else} \end{cases} .$$

Figure 2. Membership functions of the fuzzy set "approximately 8"

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Biographical Sketch

Heinrich J. Rommelfanger is Professor at the J. W. Goethe University Frankfurt/Main. He holds the chair of Mathematical Economics at the Faculty of Economics and Business Administration and is Director of the Institute of Statistics and Mathematics. He received diploma and doctoral degrees in mathematics from the University of the Saarlandes at Saarbrücken. He has done extensive research in decision theory, operations research, dynamic economic systems, financial engineering, risk management, fuzzy systems and its applications. It is a major goal of his research work to develop mathematical systems that model real problems as precise as possible and lead to convincing solutions.

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