CANONICAL FORMS FOR STATE–SPACE DESCRIPTIONS

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Contents

Summary

This chapter deals with fundamental aspects defining the structure of linear state–space models and with the representation of this structure in terms of special descriptions referred to as canonical forms. The term “structure” refers to aspects of the state–space description, which remain invariant under a variety of transformations. The set of transformations considered here are of the compensation type and include state feedback and output injection, as well as of the representation type which include state, input, and output coordinate transformations. The system structure is central to the study of systems and it is defined by a set of discrete and continuous invariants; these invariants characterize a variety of key system properties, and their type/values define the structure

of the canonical forms. The treatment of the structure is based on matrix pencils, which provide the natural operator for the different representations of the state–space model. The classical theory of Kronecker form and invariants of matrix pencils under strict equivalence is shown to be the natural vehicle that leads to the derivation of most of the canonical forms of the state–space theory. For representation transformations the theory of echelon forms of polynomial matrices provides a vehicle for introducing additional invariants for the corresponding canonical forms. Invariants and canonical forms under the general transformation group are linked to compensation theory, whereas those associated with representation transformations play a key role in system identification. Certain relations between the structure of state–space and transfer function models are also discussed.

1. Introduction

The study of problems of control analysis, control synthesis—design, and model identification, heavily relies on the notion of "system structure." The essence of system structure is that it describes aspects of the system model, which remain invariant under a set of transformations that may be applied on the system model. The system structure is a generic term that refers to system aspects such as interconnection graph for system components and functions that may be defined on the system model and referred to as system invariants; system invariants are further classified into discrete (integer, real numbers) and continuous (polynomials, rational functions etc.). The notion of structure is referred to both input–output and state–space models; the sets of transformations are clearly different for the two cases, but relationships between the structure of state–space and transfer function models exist. The notion of system structure is intimately related to that of canonical form; this corresponds to a system description, derived under the given group of transformations, expresses a specialization of the system invariants in the set of considered models, and involves the minimal number of parameters. The canonical form is thus seen as the simplest description within the equivalence class, defined by the considered transformation group that exhibits the system structure in a unique and unambiguous way.

System structure is essential for the model identification problem. Canonical forms, corresponding to representation transformations, provide a vehicle for model identification since they contain the minimal number of parameters within a given model structure to be defined. For control analysis and synthesis, aspects of the structure (as it is expressed by the system invariants) characterize the presence or absence of certain system properties; the type and values of invariants provide criteria for solvability of a number of control synthesis problems. In the area of control design, the types and values of invariants frequently impose limitations in what it is possible to achieve. Although the link between system structure and achievable performance, under certain forms of compensation, is not explicitly known, system structure expresses in a way the potential of a system to provide certain solutions to posed control problems.

This chapter considers the study of invariants and canonical forms of state–space models under the group of representation transformations, that is, state, input, output coordinate transformations and the group defined by state feedback and output injection. We consider different state–space representations, which include the autonomous and forced, and those that may, or may not, include output relations. The unifying operator for all these state–space descriptions is that of a matrix pencil. The theory of invariants and canonical forms of matrix pencils under the general group of the strict equivalence transformations is known as the Kronecker theory, and underpins the theory of invariants and canonical forms under any combination of state–space transformations. This is because any combination of state–space transformations corresponds to a subgroup of strict equivalence and thus the Kronecker invariants are also invariant under this subgroup; by considering subgroups of strict equivalence additional invariants are introduced, in addition to those of the Kronecker set. This is considered for the case of the coordinate transformation subgroup, where the additional invariants are provided by those of the echelon type canonical form of polynomial matrices.

The set of invariants and corresponding canonical forms is considered for the fundamental subgroups of the strict equivalence, and the types and values of invariants are related to corresponding system properties. We start with the autonomous description with no outputs, where the Kronecker theory is reduced to that of similarity equivalence and Jordan canonical form, then we move to the general Kronecker equivalence corresponding to the full group of the state–space transformations involving state feedback, output injection and input, state, output coordinate transformations. The Kronecker theory on the full state–space model, defined by the Rosenbrock system matrix, defines the system zero structure (see *Multivariable Poles and Zeros*). For the cases of systems with no outputs, or with no inputs, the corresponding Kronecker theory leads to canonical forms and invariants describing the controllability and observability properties respectively. Although the emphasis is on state–space descriptions, the results frequently are related to transfer function structural properties.

2. State–Space Representations, Matrix Pencils, and State–Space Transformations

Linear time invariant multivariable systems are represented in the time domain by a state variable model

$$\mathrm{S(A,B,C,D)}:\ \dot{\mathrm{x}} = \mathrm{Ax} + \mathrm{Bu},\ \ \mathrm{y} = \mathrm{Cx} + \mathrm{Du} \qquad (1)$$

where $\mathbf{x}$ is an n-vector of the state variables, $\mathbf{u}$ is an p-vector of inputs and $\mathbf{y}$ is an m-vector of outputs. $\mathbf{A, B, C, D}$ are respectively $n\times n,\ n\times p,\ m\times n,$ and $m\times p$ matrices. The above description may be represented in an autonomous or implicit form as:

$$\mathbf{S}(\boldsymbol{\Phi},\boldsymbol{\Omega}):\ \ \underbrace{\begin{bmatrix}\mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}}_{\triangleq\ \boldsymbol{\Phi}}\underbrace{\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{u}}\\ \dot{\mathbf{y}}\end{bmatrix}}_{\triangleq\dot{\boldsymbol{\xi}}} = \underbrace{\begin{bmatrix}\mathbf{A} & \mathbf{B} & \mathbf{0}\\ \mathbf{C} & \mathbf{D} & \mathbf{-I}\end{bmatrix}}_{\triangleq\ \boldsymbol{\Omega}}\underbrace{\begin{bmatrix}\mathbf{x}\\ \mathbf{u}\\ \mathbf{y}\end{bmatrix}}_{\triangleq\boldsymbol{\xi}} \qquad (2)$$

where $\boldsymbol{\Phi},\boldsymbol{\Omega}$ are coefficient matrices and $\boldsymbol{\xi}=[\mathbf{x}^t,\mathbf{u}^t,\mathbf{y}^t]^t$ is the composite vector, or implicit vector of the state–space description. The vector ξ contains the state, input and output vectors and makes no distinction between them. The $\mathbf{S=(\Phi,\Omega)}$ description belongs to the general class of generalized autonomous differential descriptions

$$\mathbf{S(F,G)}: \quad \mathbf{F\dot{z} = Gz} \tag{3}$$

where $\mathbf{F,G}$ are $r\times k$ matrices and $\mathbf{z}$ is a k-vector. The above system is characterized by the matrix pencil $\mathrm{p}\mathbf{F}-\mathbf{G}$, where $\mathrm{p}=d/dt$ denotes the derivative operator; $\mathrm{p}\mathbf{F}-\mathbf{G}$ completely characterizes the state–space description and it is referred to as implicit system pencil. Eq. (2) may also be expressed as

$$\mathbf{S}(\mathbf{\Gamma},\mathbf{\Delta}): \begin{bmatrix} \mathrm{p}\mathbf{I}-\mathbf{A} & -\mathbf{B} \\ -\mathbf{C} & -\mathbf{D} \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -\mathbf{y} \end{bmatrix}, \quad \mathbf{P}(\mathrm{p}) = \begin{bmatrix} \mathrm{p}\mathbf{I}-\mathbf{A} & -\mathbf{B} \\ -\mathbf{C} & -\mathbf{D} \end{bmatrix} \tag{4}$$

and the matrix $\mathbf{P}(\mathrm{p})$ is a matrix pencil entirely characterizing the state–space model. It is known as the *Rosenbrock system matrix pencil*. The above general state–space description may take some special forms and these are considered below:

(i) **Autonomous description with no outputs**: In this case the description has no inputs (autonomous)

$$\mathbf{S(A)}: \quad (\mathrm{p}\mathbf{I}-\mathbf{A})\,\mathbf{x}=\mathbf{0}\,, \qquad \mathbf{T(p)} = \mathrm{p}\mathbf{I}-\mathbf{A}$$

and $\mathbf{T}(\mathrm{p})$ is the internal dynamics, or pole pencil.

(ii) **Autonomous description with outputs**: In this case the description becomes

$$\mathbf{S(A,C)}: \begin{bmatrix} \mathrm{p}\mathbf{I}-\mathbf{A} \\ -\mathbf{C} \end{bmatrix}\mathbf{x} = \begin{bmatrix} \mathbf{0} \\ -\mathbf{y} \end{bmatrix}, \quad \mathbf{W}(\mathrm{p}) = \begin{bmatrix} \mathrm{p}\mathbf{I}-\mathbf{A} \\ -\mathbf{C} \end{bmatrix} \tag{5}$$

and $\mathbf{W}(\mathrm{p})$ is the *observability pencil, or state–output pencil.*

(iii) **Forced description with no outputs**: In this case the system has inputs (forced) and no outputs and the description becomes:

$$\mathbf{S(A,B)}: \; [\mathrm{p}\mathbf{I}-\mathbf{A},-\mathbf{B}]\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \mathbf{0}, \quad \mathbf{V(p)} = [\mathrm{p}\mathbf{I}-\mathbf{A},-\mathbf{B}] \tag{6}$$

and $\mathbf{V}(\mathrm{p})$ is the *controllability pencil, or input–state pencil*.

The above descriptions of the state–space type will be considered in this chapter, and it is clear that each one of them is described by a natural linear operator, which is of the matrix pencil type. The time domain descriptions, which have been considered, may be expressed in the s-domain by using Laplace transforms. In that case the matrix pencils are expressed as polynomial matrices in s.

The linear systems that are considered may be expressed in an input–output sense using the transfer function model $\mathbf{G}(s)$, where $\hat{\mathbf{y}}(s)$, $\hat{\mathbf{u}}(s)$ denote the Laplace transforms of $\mathbf{y}(t)$, $\mathbf{u}(t)$ vectors, that is $\hat{\mathbf{y}}(s) = \mathbf{G}(s)\hat{\mathbf{u}}(s)$ where $\mathbf{G}(s)$ is an $m \times p$ rational matrix. The transfer function may be also be described in a matrix fraction description form as

$$\mathbf{G}(s) = \mathbf{N}_{\mathrm{r}}(s)\, \mathbf{D}_{\mathrm{r}}(s)^{-1} = \mathbf{D}_{\mathrm{l}}(s)^{-1}\mathbf{N}_{\mathrm{l}}\ (s)$$

where $\mathbf{N}_{\mathrm{r}}(s),\ \mathbf{N}_{\mathrm{l}}(s)$ are the $m \times p$ right, left polynomial matrix numerators respectively and $\mathbf{D}_{\mathrm{r}}(s),\ \mathbf{D}_{\mathrm{l}}(s)$ are the $p \times p,\ m \times m$ polynomial matrix denominators correspondingly. It will be assumed that $\mathbf{D}_r(s),\ \mathbf{N}_r(s)$ are right coprime and $\mathbf{D}_{\mathrm{l}}(s),\ \mathbf{N}_{\mathrm{l}}(s)$ are left coprime (see *Polynomial and Matrix Fraction Description*).

On the general state–space description (1), or (4), a number of transformations may be applied, which are of the representation type (different coordinate systems) and expressed by coordinate transformations, and of the feedback compensation type; the latter is made up from state, output feedback and output injection. These transformations are denoted below and their action on the state–space model is illustrated in Figure 1. Thus, we denote:

R: $p \times p$ input coordinate transformation, $|\mathbf{R}| \neq 0$
T: $m \times m$ output coordinate transformations, $|\mathbf{T}| \neq 0$
$\mathbf{Q}, \mathbf{Q}^{-1}$: $n \times n$ pair of state coordinate transformations, $|\mathbf{Q}| \neq 0$
F: $m \times p$ constant output feedback matrix.
L: $n \times p$ state feedback matrix
K: $n \times m$ output injection matrix

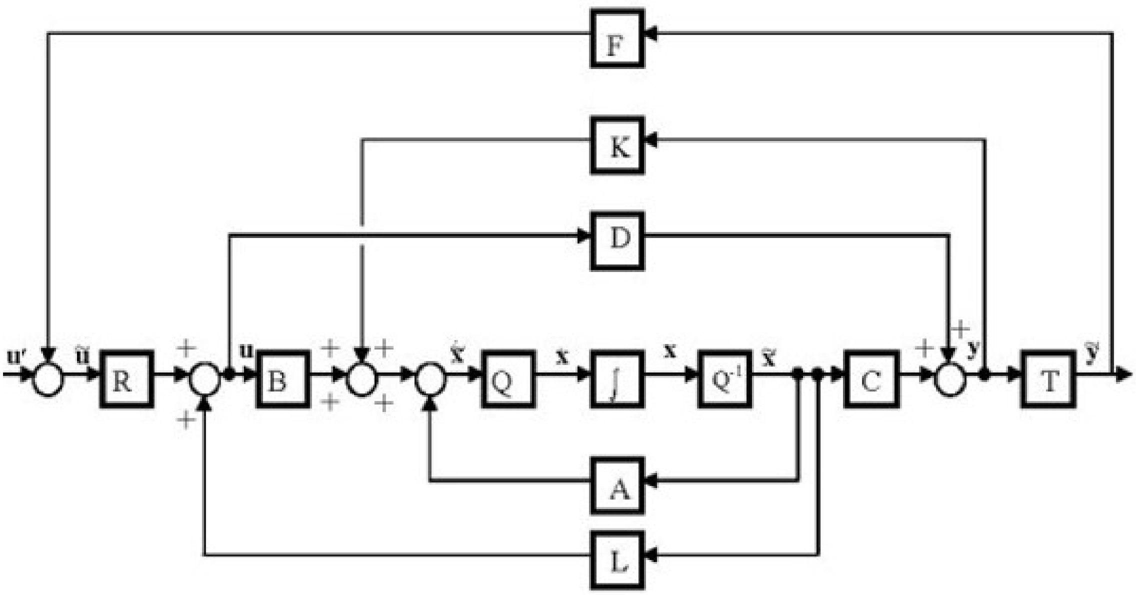

Figure 1: State–space transformation

The above set of transformations $(\mathbf{R}, \mathbf{T}, \mathbf{Q}^{-1}, \mathbf{L}, \mathbf{K})$ when applied on the original system described by $\mathbf{P}(s)$, produce a system $\mathbf{S}'(\mathbf{A}', \mathbf{B}', \mathbf{C}', \mathbf{D})$ described by a $\mathbf{P}'(s)$ system matrix where

$$\mathbf{P}'(s) = \begin{bmatrix} \mathbf{Q}^{-1} & \mathbf{K} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \begin{bmatrix} s\mathbf{I} - \mathbf{A} & -\mathbf{B} \\ -\mathbf{C} & -\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{L} & \mathbf{R} \end{bmatrix} \tag{7}$$

Clearly, $\mathbf{P}(s)$, $\mathbf{P}'(s)$ are related by a certain form of equivalence, which is defined on matrix pencils. Such transformations are referred to as the full set of state–space transformation, or as the Kronecker set of transformations, for reasons that will become clear subsequently. For the representations $\mathbf{S(A)}, \mathbf{S(A,C)}, \mathbf{S(A,B)}$ the Kronecker set of transformations specializes as shown next:

(i) $\mathbf{S(A)}$ description: Then the transformations on the $\mathbf{T}(s)$ pencils are similar (state coordination type) i.e.

$$\mathbf{T}'(s) = \mathbf{Q}^{-1}(s\mathbf{I} - \mathbf{A})\mathbf{Q} \tag{8}$$

(ii) $\mathbf{S(A,C)}$ description: Then the transformations on $\mathbf{W}(s)$ pencil are of the output injection and state, output coordination, type, i.e.,

$$\mathbf{W}'(s) = \begin{bmatrix} \mathbf{Q}^{-1} & \mathbf{K} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \begin{bmatrix} s\mathbf{I} - \mathbf{A} \\ -\mathbf{C} \end{bmatrix} \mathbf{Q} \tag{9}$$

(iii) $\mathbf{S(A,B)}$ description: Then the transformations on $\mathbf{V}(s)$ pencil are of the state feedback, input, state coordination type, i.e.,

$$\mathbf{V}'(s) = \mathbf{Q}^{-1}\left[s\mathbf{I} - \mathbf{A}, -\mathbf{B}\right] \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{L} & \mathbf{R} \end{bmatrix} \tag{10}$$

Conditions (7), (8), (9), and (10) express the action of transformation groups on matrix pencil descriptions of state–space models. These transformation groups are by no means exhaustive, and subgroups of them may be defined using the above descriptions. The study of invariants and canonical forms clearly relies on the theory of matrix pencils which is briefly summarized below.

3. Matrix Pencils and Kronecker Form

3.1. Background

The development of the theory of matrix pencil invariants and canonical forms requires the introduction of some basic results that may be found in algebra books, as well as some definitions on a number of important notions.

Definition 1: Let $\mathcal{X}$ be a set, $\mathcal{E}$ be an equivalence relation on $\mathcal{X}$ and let $x \in \mathcal{X}$; the equivalence class, or orbit of x under $\mathcal{E}$ is defined by $\mathcal{E}(x) = \{y : y \in \mathcal{X} : x\mathcal{E}y\}$. The set of all equivalence classes is called the *quotient set*, or *orbit set* and it is denoted by $\mathcal{X}/\mathcal{E}$.

Theorem 1: If $\mathcal{E}$ is an equivalence relation on a set $\mathcal{X}$, then the family of all $\mathcal{E}$ - equivalence classes forms a partition of $\mathcal{X}$, i.e., there exist $\{x_i \in \mathcal{X} : i = 1, 2, \ldots\}$ such that

$$\mathcal{X} = \mathcal{E}(x_1) \cup \ldots \cup \mathcal{E}(x_i) \ \ \ldots, \qquad \mathcal{E}(x_i) \cap \mathcal{E}(x_j) \neq \emptyset \tag{11}$$

The set of elements $\{x_i,\ i = 1,\ 2, \ldots\}$ for which conditions (11) hold true is called a *system of distinct representatives* for $\mathcal{E}$ and it is a subset $\mathcal{D}$ of $\mathcal{X}$ that contains precisely one element from each of $\mathcal{E}$ - equivalence classes.

Definition 2: Let $\mathcal{X}$, $\mathcal{T}$ be sets, $\mathcal{E}$ an equivalence relation defined on $\mathcal{X}$. We define:

(i) a function $f : \mathcal{X} \to \mathcal{T}$ is called an *invariant* of $\mathcal{E}$, when $\forall x, y\ \mathcal{X}$: $x\ \mathcal{E}\ y$ implies $f(x) = f(y)$,

(ii). $f : \mathcal{X} \to \mathcal{T}$ is called a *complete invariant* for $\mathcal{E}$, when $f(x) = f(y)$ implies $x\ \mathcal{E}\ y$,

(iii) a set of invariants $\{f_i : f_i : \mathcal{X} \to \mathcal{T}_i,\ i = 1,\ 2, \ldots, k\}$ is a complete set for $\mathcal{E}$, if the map f defined by $f : \mathcal{X} \to \mathcal{T}_1 \times \ldots \times \mathcal{T}_k$, where $x \to (f_1(x), \ldots, f_k(x))$ is a *complete invariant* for $\mathcal{E}$ on $\mathcal{X}$.

A complete invariant defines a one to one correspondence between the equivalence classes $\mathcal{E}(x)$ and the image of f. If $f : \mathcal{X} \to \mathcal{T}_1 \times \ldots \times \mathcal{T}_k$ where $x \to (f_1(x), \ldots, f_k(x))$ is a complete invariant for $\mathcal{E}$ on $\mathcal{X}$, then the set $(f_1(x), \ldots, f_k(x))$ characterizes uniquely $\mathcal{E}(x)$. The values $f_i(x)$ are often called invariants and this definition will be used later in the chapter. If we "specialize" the invariant f such that its image $\mathcal{T} \subset \mathcal{X}$, expresses f in the "simplest" possible way, then we define the notion of a canonical element, or canonical form.

Definition 3: A set of *canonical forms* for $\mathcal{E}$ equivalence on $\mathcal{X}$ is a subset $\mathcal{C}$ of $\mathcal{X}$ such that $\forall x \in \mathcal{X}$ there is a unique $c \in \mathcal{C}$ for which $x\ \mathcal{E}\ c$.

Equivalence relations may be introduced by the action of groups on sets and in particular by the action of transformation groups. If $\mathcal{X}$ is a set and $\mathcal{G}(\mathcal{X})$ is the group of permutations of $\mathcal{X}$, any subgroup $\mathcal{H}(\mathcal{X})$ of $\mathcal{G}(\mathcal{X})$ is called a transformation group of $\mathcal{X}$. A transformation group is therefore a set $\mathcal{H}$ of mappings of $\mathcal{X}$ into $\mathcal{X}$, which are bijective. If $(\mathcal{G},*)$ is a group operating on a set $\mathcal{X}$, where ***** denotes the group operation and o the operation of $\mathcal{G}$ on $\mathcal{X}$, then the relation: there exists $s \circ \mathcal{G}$ such that $y = s \circ x$, is an equivalence relation and it is called the induced equivalence relation of $\mathcal{G}$ on $\mathcal{X}$. $\mathcal{G} \circ x$ is then the equivalence class of x under this relation. The above are considered next in the case of special sets associated with matrix pencils.

3.2. Matrix Pencils and Strict Equivalence

Consider the set of ordered pairs of matrices $\mathcal{L} = \{\mathbf{L} : \mathbf{L} = (\mathbf{F}, -\mathbf{G}), \mathbf{F}, \mathbf{G} \in \mathbb{R}^{r\times q}\}$ of $r \times q$ dimension and let $\theta = (s, \hat{s})$ be an ordered pair of indeterminates. The polynomial matrix $\mathbf{L}(s, \hat{s}) = s\mathbf{F} - \mathbf{G} \in \boldsymbol{R}^{r\times q}[s, \hat{s}]$ is called the *homogeneous matrix pencil* of $\mathbf{L} \in \mathcal{L}$, $\mathbf{L}(s) = s\mathbf{F} - \mathbf{G} \in \boldsymbol{R}^{r\times q}[s]$ is the *prime* and $\mathbf{L}(\hat{s}) = \mathbf{F} - \hat{s}\mathbf{G} \in \boldsymbol{R}^{r\times q}[\hat{s}]$ is the *dual pencil* of $\mathcal{L}$. If $r = q$ and $\det(s\mathbf{F} - \hat{s}\mathbf{G}) \neq 0$ (for all $(s, \hat{s})$, then the pencil is called *regular*; otherwise, i.e. $r \neq q$, or $r = q$ and $\det(s\mathbf{F} - \hat{s}\mathbf{G}) \equiv 0$ (for all $(s, \hat{s})$, then the pencil is called *singular*. For every $\mathbf{L} \in \mathcal{L}$ the matrix $[\mathbf{L}] = [\mathbf{F}, -\mathbf{G}] \in \mathbb{R}^{r\times 2q}$ will be referred to as the matrix representation of $\mathbf{L}$.

The properties of matrix pencils are considered under certain forms of equivalence, which are introduced by the action of transformation groups. Such groups and their actions on pencils are defined as follows: consider first the set $\mathcal{K} = \{\mathbf{K}: \mathbf{K} = (\mathbf{M},\mathbf{N}), \mathbf{M} \in \mathcal{F}^{r\times r}, \mathbf{N} \in \mathcal{F}^{2q\times 2q}, |\mathbf{M}|, |\mathbf{N}| \neq 0\}$, where $\mathcal{F}$ is $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{R}[s]$, $\boldsymbol{R}[\hat{s}]$ and a composition rule (*) defined on $\mathcal{K}$ as : $* \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ where for all $\mathbf{K}_1 = (\mathbf{M}_1, \mathbf{N}_1)$, $\mathbf{K}_2 = (\mathbf{M}_2, \mathbf{N}_2) \to \mathcal{K}$ then

$$\mathbf{K}_1 * \mathbf{K}_2 \triangleq (\mathbf{M}_1, \mathbf{N}_1) * (\mathbf{M}_2, \mathbf{N}_2) = (\mathbf{M}_1 \mathbf{M}_2, \mathbf{N}_2 \mathbf{N}_1) \tag{12}$$

It may be readily verified that $(\mathcal{K}, *)$ is a group with identity element $(\mathbf{I}_r, \mathbf{I}_{2q})$. The action of $\mathcal{K}$ on $\mathcal{L}$ is defined by $\circ: \mathcal{K} \times \mathcal{L} \to \mathcal{L}$: $\forall \mathbf{K} = (\mathbf{M}, \mathbf{N}) \in \mathcal{K}$ and for $\mathbf{L} = (\mathbf{F}, -\mathbf{G}) \in \mathcal{L}$, then

$$\mathbf{K} \circ \mathbf{L} \triangleq \mathbf{L}' = (\mathbf{F}', -\mathbf{G}') = \mathbf{M}\, [\mathbf{F}, -\mathbf{G}]\, \mathbf{N} \tag{13}$$

Such an action defines an equivalence relation $\mathcal{E}_{\mathcal{K}}$ on $\mathcal{L}$ and $\mathcal{E}_{\mathcal{K}}(\mathbf{L})$ denotes the corresponding equivalence class. Some important subgroups of $\mathcal{K}$ and their corresponding equivalence are described below:

Definition 4: The subgroup $(\mathcal{H},*)$ of $(\mathcal{K},*)$, where

$$\mathcal{H} = \{\mathbf{H} : \mathbf{H} = (\mathbf{R}, \mathbf{T}), \mathbf{R} \in \mathbb{R}^{r\times r}, \mathbf{T} = \text{diag}\,\{\mathbf{Q}, \mathbf{Q}\} \in \mathbb{R}^{2q\times 2q}, |\mathbf{R}|, |\mathbf{Q}| \neq 0\} \tag{14}$$

is called the *strict-equivalence group* (SEG). The action of $\mathcal{H}$ on $\mathcal{L}$ is defined by $\circ: \mathcal{H} \times \mathcal{L} \to \mathcal{L}: \forall \mathbf{H} \in \mathcal{H}$ and for $\mathbf{L} = (\mathbf{F}, \mathbf{G}) \in \mathcal{L}$ then

$$\mathbf{H} \circ \mathbf{L} \triangleq \mathbf{L}' = \mathbf{R}\, [\mathbf{F}, -\mathbf{G}]\, \mathbf{T} = [\mathbf{R}\,\mathbf{F}\,\mathbf{Q}, -\mathbf{R}\,\mathbf{G}\mathbf{Q}] \tag{15}$$

which on the corresponding pencils $\mathbf{L}(s) = s\mathbf{F}\text{-}\mathbf{G}$, $\mathbf{L}'(s) = s\mathbf{F}'\text{-}\mathbf{G}'$ expresses the relation

$$\mathbf{L}'(s) = s\mathbf{F}' - \mathbf{G}' = \mathbf{R}\,(s\mathbf{F} - \mathbf{G})\,\mathbf{Q} = \mathbf{R}\,\mathbf{L}(s)\,\mathbf{Q} \tag{16}$$

This equivalence relation $\mathcal{E}_{\mathcal{H}}$ defined on $\mathcal{L}$ is called *strict equivalence* and the corresponding class is denoted by $\mathcal{E}_{\mathcal{H}}(\mathbf{F},\mathbf{G})$

The classical theory of matrix pencils is based on the study of invariants and canonical forms under strict equivalence. Such a theory is predominantly algebraic and it is based on invariants introduced by Smith forms and minimal bases of rational vector spaces. There is, however, a geometric dimension linked to the fundamentals of geometric theory of linear systems and in particular to the characterization of families of invariant subspaces. Computations are reduced to those of the generalized eigenvalue–eigenvector type and related problems of the Kronecker canonical form. The classical analysis is based on the pencil $s\mathbf{F}-\mathbf{G}$, or on its homogeneous form $s\mathbf{F}-\hat{s}\mathbf{G}$; the results of strict equivalence, however, may be interpreted on the ordered pair of matrices $(\mathbf{F},\mathbf{G})$. Such interpretation allows the characterization of invariants in terms of number theoretic properties based on Toeplitz matrices. The classical algebraic characterization will be summarized in the following.

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